

The Connection between the Number of Realizations for Degree Sequences and Majorization

Annabell Berger¹

Department of Computer Science, Martin-Luther-Universität Halle-Wittenberg, Germany

Abstract

We consider three different realization problems for degree sequences. For the undirected case, the *graph realization problem* is to find for a given integer sequence $s := (a_1, \dots, a_n)$ a labeled graph $G = (V, E)$ (without loops and parallel edges) such that we have for the vertex degrees $d(v_i) = a_i$. For the directed case, the *loop-digraph realization problem* or the *digraph realization problem* is to find for a given sequence $S := \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n}$ a) a labeled digraph $G = (V, A)$ (without parallel arcs) with at most one loop per vertex or b) a labeled digraph $G = (V, A)$ (without parallel arcs and loops) such that we have for the vertex indegrees $d^-(v_i) = a_i$ and for the vertex outdegrees $d^+(v_i) = b_i$. The loop-digraph realization problem can also be seen as the *bipartite undirected graph realization problem*. The classic literature by Erdős (1960), Gallai (1960), Fulkerson (1960), Ryser (1957), Gale (1957), Chen (1966) and Anstee (1982) provides characterizations for the existence of such realizations which amount to the evaluation of at most n different inequalities. This approach is strongly related to the concept of majorization (Hardy, Littlewood and Polya (1934)). Mahadev and Peled (1995) extended the classic approach to a more general result for graphs, i.e. an integer sequence s is realizable as a graph if and only if there exists a sequence s' which is realizable and majorizes s . This approach also leads to an efficient algorithm constructing a sequence of graph sequences $s' =: s_1, \dots, s_k =: s$ where each s_i majorizes s_{i+1} . This view gives a very short and simple proof for the Erdős-Gallai Theorem. We extend this approach to the loop-digraph realization problem and the digraph realization problem. The main result of our paper is to show a connection between majorization and the number of realizations for an (integer) sequence. In all three cases we show for two (integer) sequences S', S where S' majorizes S in a certain sense, that S possesses more realizations than S' . For the first time, we give a precise characterization which (integer) sequences possess the most realizations for fixed n and fixed number of arcs $m := \sum_{i=1}^n a_i$. These are regular (integer) sequences when n is a factor of m and so-called *minconvex sequences* (graphs, loop-digraphs) or *opposed minconvex sequences* (digraphs) otherwise.

Keywords: degree sequence, Gale-Ryser theorem, Erdős-Gallai theorem, majorization, minconvex sequence, opposed sequence, graphical sequence

Email address: berger@informatik.uni-halle.de (Annabell Berger)

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1. Introduction

Realization problems. In our paper we treat with three related problems in the topic of realizing degree sequences.

Problem 1 (digraph realization problem/loop-digraph realization problem). *Given is a finite sequence $S := \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n}$ with $a_i, b_i \in \mathbb{Z}_0^+$. Does there exist*

- 1) *a digraph $G = (V, A)$ without parallel arcs and loops*
- 2) *a digraph $G = (V, A)$ without parallel arcs and at most one loop per vertex*

with a labeled vertex set $V := \{v_1, \dots, v_n\}$ such that we have indegree $d_G^-(v_i) = a_i$ and outdegree $d_G^+(v_i) = b_i$ for all $v_i \in V$?

If the answer is “yes”, we call sequence S in case 1) *digraph sequence* and in case 2) *loop-digraph sequence*. Analogously, we call digraph G a *digraph realization* or *loop-digraph realization*, respectively. We exclude tuples $\binom{0}{0}$ in S . In case 1) we demand $0 \leq a_i, b_i \leq n - 1$ and in case 2) $0 \leq a_i, b_i \leq n$. Furthermore, we will tacitly assume that $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, as this is obviously a necessary condition for any realization to exist, since the number of ingoing arcs must equal the number of outgoing arcs. It is possible to reduce the loop-digraph realization problem to a *bipartite realization problem*. Namely, one only has to consider the adjacency matrix of a loop-digraph realization as the adjacency matrix of an undirected bipartite graph. The a_i and b_i correspond to the vertex degrees of the two independent vertex sets. Furthermore, we consider the *graph realization problem*. Here, we have to decide whether there exists for a given integer sequence $s := (a_1, \dots, a_n)$ a labeled graph $G = (V, E)$ such that the vertex degree of vertex v_i matches the number a_i for each $i \in \mathbb{N}_n$. Analogously to the directed case, we call such a sequence *graph sequence* and the graph G *graph realization*. We demand for integer sequences that $\sum_{i=1}^n a_i$ is even, because this is an obvious condition for realizability. There is a very simple connection between the graph realization problem and the digraph realization problem. For a given integer sequence $s := (a_1, \dots, a_n)$ one only has to construct sequence $S := \binom{a_1}{a_1}, \dots, \binom{a_n}{a_n}$ and to solve the digraph realization problem. A result of Chen [Che80] shows that each digraph sequence of this form also possesses a symmetric digraph realization $G = (V, A)$, i.e. for each arc $(v, w) \in A(G)$ there also exist an arc $(w, v) \in A(G)$. Replacing each arc pair $(v, w), (w, v)$ by an edge $\{v, w\}$ results in a graph realization for integer sequence s . In the classical literature we observe two different approaches to solve these realization problems, i.e. a recursive algorithm constructing a realization as developed by Havel and Hakimi [Hav55, Hak65] for the undirected case by Kleitman and Wang [KW73] for the directed case. The so-called *characterization of sequences* is the second approach which lastly results in the evaluation of at most n inequalities. Our work is strongly connected with this approach. For that reason we briefly survey these results in this section. There exist several different notions for these results. We decided to use the classical connection to the relation *majorization* \prec of real vectors which was introduced by Hardy, Littlewood and Polya 1934 [HLP34]. In our work we identify an integer sequence $s := (a_1, \dots, a_n)$ with an n -dimensional integer vector and a sequence $S := \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n}$ with a pair of n -dimensional integer vectors $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. For simplicity, we often denote a sequence $S := \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n}$ by $S = \binom{a}{b}$. Furthermore, we need a notion for a lexicographical

sorting of a sequence. We call the relation $\geq_{lex} \subset \mathbb{Z}^+ \times \mathbb{Z}^+$ with

$$\begin{pmatrix} a \\ b \end{pmatrix} \geq_{lex} \begin{pmatrix} a' \\ b' \end{pmatrix} \Leftrightarrow a > a' \text{ or } (a = a' \wedge b > b'),$$

lexicographical relation. The lexicographical ordering is a total ordering, i.e., reflexive, antisymmetric and transitive. Hence, it is possible to number all tuples of a sequence S such that we have $\begin{pmatrix} a_i \\ b_i \end{pmatrix} \geq_{lex} \begin{pmatrix} a_j \\ b_j \end{pmatrix}$ if and only if $i \leq j$. We denote such a labeling of a sequence by *decreasing lexicographical sorting*.

Definition 1.1 (Majorization). *We define the relation majorization $\prec \subset \mathbb{Z}^n \times \mathbb{Z}^n$ by $a \prec a'$ if and only if*

$$\begin{aligned} \sum_{i=1}^k a_i &\leq \sum_{i=1}^k a'_i \text{ for all } k \in \{1, \dots, n-1\} \text{ and} \\ \sum_{i=1}^n a_i &= \sum_{i=1}^n a'_i. \end{aligned}$$

Vector a is said to be majorized by a' or a' majorizes a , respectively.

Next we consider *threshold sequences* for graph sequences, loop-digraph sequences and digraph sequences. Threshold sequences are graph sequences (loop-digraph sequences, digraph sequences) which only possess one unique graph realization (loop-digraph realization, digraph realization). For more details about threshold sequences in the undirected case consider the book of Mahadev and Peled [MP95]. Note, that a threshold sequence in the undirected case can only majorize other sequences but cannot be majorized by an other sequence. Each realizable (integer) sequence has at least one corresponding threshold sequence which can easily be constructed by a *Ferrers matrix*. A Ferrers matrix corresponds to exactly one threshold sequence. Moreover, a Ferrers matrix is the adjacency matrix of the one unique realization of a threshold sequence. In the directed case, the sequence of row and column sums are exactly the first and second components of the threshold sequence. The interpretation in the undirected case is very simple, if one uses the view on digraph sequences $S := \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ a_n \end{pmatrix}$. Ferrers introduced this notion in the context of partitions (rows and column sums can be interpreted as a partition and its dual partition). For more information see Sylvester 1882 [SF82] or visit the overview about Norman Macleod Ferrers in [Fer]. In our context, we determine for a given sequence a corresponding threshold sequence. In a certain sense such a sequence S is majorized by its threshold sequence S' if and only if we have a realizable sequence S . In the directed case we only compare one component of both sequences S and S' with respect to majorization. Indeed, this is the general statement of the characterization approach for all of our realization problems. For the undirected case, we again switch to the digraph realization problem for sequence $S := \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ a_n \end{pmatrix}$. Definition 1.1 of majorization implies that we have to evaluate n inequalities. Clearly, we have different Ferrers matrices for the loop-digraph and digraph realization problem. We formally introduce these matrices and present the classical characterization results with this notion.

Definition 1.2 (Ferrers matrix for loop-digraphs, threshold sequence for loop-digraphs). *Given is a sequence $S := \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ with $a_i \geq a_{i+1}$ for all $i \in \{1, \dots, n-1\}$. We*

construct an $(n \times n)$ -matrix $F := (F_{ij})_{i,j \in \mathbb{N}_n}$ with

$$F_{ij} := \begin{cases} 1 & \text{if } j \leq b_i \\ 0 & \text{if } j > b_i. \end{cases}$$

Let a'_i denote the i th column sum of F . We call sequence $S' := (a'_1), \dots, (a'_n)$ a corresponding threshold loop-digraph sequence for sequence S .

Clearly, each sequence (taking our definition with $0 \leq b_i \leq n$) possesses such a Ferrers matrix. We cite the classical characterization result from Gale and Ryser [Gal57, Rys57] loop-digraphs or bipartite graphs, respectively.

Theorem 1 (Gale,Ryser). *Let $S := (a_i)$ be a sequence with $a_i \geq a_{i+1}$ for all $i \in \{1, \dots, n-1\}$ and $S' = (a'_i)$ be a corresponding threshold loop-digraph sequence. Sequence S is a loop-digraph sequence if and only if $a \prec a'$.*

Note, that although a sequence S always has a threshold sequence, it has not to be a loop-digraph sequence. Next we give the characterization result for digraphs. Again, we introduce Ferrers matrices in this context.

Definition 1.3 (Ferrers matrix for digraphs, threshold sequence for digraphs). *Given is sequence $S := (a_i)$ with $a_i \geq a_{i+1}$ for all $i \in \{1, \dots, n-1\}$. We construct an $(n \times n)$ -matrix $F := (F_{ij})_{i,j \in \mathbb{N}_n}$ with*

$$F_{ij} := \begin{cases} 1 & \text{if } (j \leq b_i \wedge j < i) \vee (j \leq b_i + 1 \wedge j > i) \\ 0 & \text{otherwise.} \end{cases}$$

Let a'_i denote the i th column sum of F . We call sequence $S' := (a'_1), \dots, (a'_n)$ a corresponding threshold digraph sequence for sequence S .

In contrast to the analogous loop-digraph realization problem, a sequence S can possess several different corresponding threshold digraph sequences. The reason is that different sortings of the tuples in S with respect to decreasingly sorted a_i can lead to different Ferrers matrices. Nevertheless, the characterization theorem is true for an arbitrarily chosen threshold digraph sequence. This result is relatively unknown until today. In 1966 Chen [Che66] only proved the characterization theorem for lexicographic decreasingly sorted sequences (and its Ferrers matrices). Very recently, we proved this theorem in its general form in [Ber11]. Subsequently we found out that Anstee already observed this in 1982 in the context of bipartite $(0, 1)$ -matrices with several restrictions [Ans82].

Theorem 2 (Fulkerson,Chen,Anstee). *[Ful60, Che66, Ans82] Let $S := (a_i)$ be a sequence with $a_i \geq a_{i+1}$ for all $i \in \{1, \dots, n-1\}$ and $S' = (a'_i)$ be the corresponding threshold digraph sequence. Sequence S is a digraph sequence if and only if $a \prec a'$.*

Applying this theorem to sequence $S := (a_1), \dots, (a_n)$ and keeping in mind that each digraph sequence of this form also possesses a symmetric digraph realization [Che66] we get the well-known result of Erdős and Gallai. First, we define the *threshold graph sequence* $s' := (a'_1, \dots, a'_n)$ of $s = (a_1, \dots, a_n)$ to be the first component of threshold digraph sequence $S' := (a'_1), \dots, (a'_n)$ from Definition 1.3.

Theorem 3 (Erdős, Gallai 1960). [EG60] Let $s = (a_1, \dots, a_n)$ be an integer sequence with $a_i \geq a_{i+1}$ for all $i \in \{1, \dots, n-1\}$ and s' be the corresponding threshold graph sequence. Integer sequence s is a graph sequence if and only if $s \prec s'$.

Note, that an integer sequence can possess a threshold sequence but is not a graph sequence. Consider for example integer sequence $(3, 3, 1, 1)$. To the best of our knowledge the complexity status for the corresponding counting problems, i.e. counting the number of realizations for a given (integer) sequence, is open for all three realization problems. In contrast, for a related problem —the bipartite multigraph realization problem, i.e. counting the number of bipartite multigraphs for a given sequence, is known to be $\#P$ -hard [DKM95]. On the other hand there exist a lot of work for counting graph realizations and loop-digraph realizations to a given (integer) sequence in the context of approximation, approximation algorithms and randomized algorithms see for example [BC78, MW91, JSV04, BBV06, BLSY10, Bar08, CGM08]. However, in our work we concentrate on a connection between different (integer) sequences with respect to the number of their realizations.

Our Contribution. There are different approaches to prove Theorems 3, 2 and 1 using for example network flow theory or induction. In the next section we give generalizations for these theorems. The first one for graphs is from Mahadev and Peled [MP95]. We follow their approach and extend it to the two directed cases. The proofs are very simple and can easily be used for proving the last three given theorems. Additionally, they give efficient algorithms for constructing a suitable realization. We believe these are the shortest known proofs for the characterization of degree sequences. Our main question for this work was to find a relationship between different sequences of n tuples and m arcs with respect to the number of realizations of them. It is well-known that threshold sequences possess exactly one unique realization. We found a regular connection which is in a certain sense a generalization of this insight. In particular we found that for two sequences S, S' where S' majorizes S in a certain sense, the number of graph realizations of S is larger than the number of realizations of S' . This we could prove for all three realization problems. Clearly, there are several restrictions. So, for the loop-digraph problem the first component of S has to be decreasingly sorted, for the digraph problem S has to be lexicographical decreasingly sorted, and in the graph realization problem the integers are decreasingly sorted. Note, that it is sometimes possible to find a lower bound on the number of realizations for sequences with exponentially many realizations. We do not explicitly point out this result, but it can easily be computed in determining a possibly long transfer path, i.e. a sequence of sequences S^1, \dots, S^k with a special type of transfers. Such a result can be very important for the well-known *uniform sampling problem*. In the topic of network analysis people often try to sample a realization for a given sequence uniformly at random. Clearly, the mixing time also depends on the number of existent realizations. If there only exists a polynomial number of realizations one could enumerate all of these realizations and solving the problem by choosing a number uniformly at random. Our approach leads to several first insights with respect to this problem and in some cases one can easily determine that there has to be exponentially many of such realizations. Lastly, we proved for all three problems that so-called minconvex sequences possess the largest number of realizations under all realizations with fixed n and m . For digraph sequences the result is more complicated. Here, a special kind of minconvex

sequences possess the largest number of realizations, i.e. opposed sequences where the first component is in decreasing and the second component is in increasing order. Minconvex sequences are in certain sense the ‘contrary threshold sequences’. They can only be majorized (and not majorize) by sequences and possess the largest number of realizations.

Overview. In Section 2, we generalize the characterization Theorems 1 and 2. The proofs lead to a new type of realization algorithms. In Section 3, we explore the connection between majorization and the number of realizations for a given degree sequence. Furthermore, we show that minconvex sequences possess the largest number of realizations.

2. Generalizations of Characterizations of Degree Sequences

Similar but not identical to Mahadev/Peled [MP95] and Marshall/Olkin [MO79] we define *transfers* on integer sequences. We call integer sequences $e_i := (e_{i1}, \dots, e_{in})$ with $e_{ii} = 1$ and $e_{ik} = 0$ for all $k \in \mathbb{N}_n \setminus \{i\}$ *unit sequences*.

Definition 2.1 (transfer). *Let $s := (a_1, \dots, a_n)$ be an integer sequence with $a_i \geq a_j + 2$ for some $i, j \in \{1, \dots, n\}$ with $i < j$. An operation $t_{i,j}(s) := s - e_i + e_j$ on the set of integer sequences, where e_i and e_j are unit sequences, is called an (i, j) -transfer. Sometimes, we use the notion transfer.*

We repeat a classical result of Muirhead [Mui02] in a version by Mahadev and Peled [MP95] with a small distinction for our specific investigations. This proof gives also an instruction how to construct an integer sequence $s := (a_1, \dots, a_n)$ by another integer sequence $s' := (a'_1, \dots, a'_n)$, when s is majorized by s' .

Theorem 4 (Muirhead 1902). *[Mui02] Let $s := (a_1, \dots, a_n)$ with $a_1 \geq \dots \geq a_n$ and $s' := (a'_1, \dots, a'_n)$ be integer sequences and $s \prec s'$. Then s can be derived from s' by successive application of a finite number k of (i_l, j_l) -transfers, i.e. $s = t_{i_k, j_k} \circ t_{i_{k-1}, j_{k-1}} \circ \dots \circ t_{i_1, j_1}(s')$.*

Proof. We can assume that a and a' are different. Let l be the largest integer for which

$$\sum_{i=1}^l a_i < \sum_{i=1}^l a'_i.$$

Then $a_{l+1} > a'_{l+1}$, and there is a largest integer $k \leq l$ for which $a_k < a'_k$. Thus, $a'_k > a_k \geq a_{l+1} > a'_{l+1}$. It follows

$$a'_k \geq a'_{l+1} + 2. \quad (*)$$

Let $s'' := t_{k, l+1}(s')$ be obtained from s' by a $(k, l+1)$ -transfer. Clearly, $\sum_{i=1}^r a''_i \geq \sum_{i=1}^r a_i$ holds for all $r < k$ and $r > l$. Assume this inequality failed for some $r \in \{k, \dots, l\}$, namely $\sum_{i=1}^r a''_i < \sum_{i=1}^r a_i$. By our assumption with respect to the choice of k and l we have $a_{r+1} \geq a'_{r+1} = a''_{r+1}$ for all r with $k \leq r \leq l-1$. Hence, we find $\sum_{i=1}^l a''_i < \sum_{i=1}^l a_i$ if we add all components for indices $r+1$ to l on both sides of the inequality. On the other hand, we know that $a_{l+1} > a'_{l+1} = a''_{l+1} - 1$. It follows $\sum_{i=1}^{l+1} a'_i = \sum_{i=1}^{l+1} a''_i < \sum_{i=1}^{l+1} a_i$, which contradicts the fact of equality. Hence, $s \prec s'' \prec s'$. Repeating this approach proves the theorem. \square

In contrast to the original proof we have only ordered the integers in s whereas we omitted the ordering of the integers of s' . This is in fact very important with respect to the digraph realization problem. It turns out that Muirhead's Lemma can be used for the construction of a graph realization, loop-digraph realization and a digraph realization if one starts with the corresponding threshold sequence of a given graph sequence, loop-digraph sequence or digraph sequence, respectively. The variant for graph sequences was given by Mahadev and Peled [MP95].

Theorem 5 (Mahadev and Peled 1995). *If s' is a graph sequence and s is an integer sequence such that $s \prec s'$, then there exist graph sequences*

$$s' := s^1, s^2, \dots, s^r =: s$$

such that s^i yields s^{i+1} by a unit transfer, i.e. $s^{i+1} \prec s^i$. In particular, s is a graph sequence.

Mahadev and Peled give a proof which uses Theorem 4 by Muirhead, constructing for each sequence step by step a graph realization. Moreover, we can consider this theorem as a generalization of Theorem 3 by Erdős and Gallai, since it is very easy to determine a threshold graph sequence s' for an integer sequence s by using the Ferrers matrix of Definition 1.3. On the other hand, we get a further realization algorithm for a given graph sequence. In the following, we want to extend this approach to loop-digraph sequences and digraph sequences.

Theorem 6. *Let $S' := \binom{a'}{b}$ be a digraph sequence and $S := \binom{a}{b}$ be a sequence such that $a \prec a'$ and $a_1 \geq \dots \geq a_n$. Then there exist digraph sequences $S' := \binom{a^1}{b}, \binom{a^2}{b}, \dots, \binom{a^r}{b} =: S$ such that a^i yields a^{i+1} by a unit transfer, i.e. $a^{i+1} \prec a^i$. In particular, S is a digraph sequence.*

Proof. The existence of integer sequences $a^1 := a', \dots, a^r := a$ such that a^i yields a^{i+1} by a unit transfer follows from Theorem 4 by Muirhead. We show by induction with respect to the length r of the sequence of integer sequences a^i that each $\binom{a^i}{b}$ is a digraph sequence. Let G^i be a digraph realization of sequence $S^i = \binom{a^i}{b}$ and let $S^{i+1} = \binom{a^{i+1}}{b}$ obtained by a (k, l) -transfer, i.e. $a^{i+1} = t_{k,l}(a^i)$. With (*) in the proof of Theorem 4, we have $a_k^i \geq a_l^i + 2$. Hence, there exist two different vertices $j, j' \in V(G^i)$ with $j \neq k$, $j' \neq k$ where $(j, k), (j', k) \in A(G^i)$ and $(j, l), (j', l) \notin A(G^i)$. Clearly, for at least one of the vertices j and j' we find $j \neq l$ or $j' \neq l$. W.l.o.g. we assume $j \neq l$. Then this digraph G^{i+1} is defined to possess the arc set $A(G^{i+1}) := (A(G^i) \setminus \{(j, k)\}) \cup \{(j, l)\}$ is a digraph realization of sequence S^{i+1} . \square

Remark 1. *In Theorem 6 it is not necessary to sort digraph sequence S if $t_{i,j}(a') = a$, i.e. $r = 2$. The reason is that this sorting is only necessary for Muirhead's Theorem 4. The construction of a digraph does not need this sorting.*

Clearly, this theorem generalizes Theorem 2 of Fulkerson, Chen and Anstee if we demand that S' is a threshold digraph sequence of S . Furthermore, we get a new digraph realization algorithm for S , which first determines for a given sequence $S = \binom{a}{b}$ its threshold sequence $S' := \binom{a'}{b}$. In the case that S is a digraph sequence ($a \prec a'$),

the algorithm determines with Theorem 4 and the proof of Theorem 6 step by step a sequence of digraph sequences $S' := \binom{a^1}{b}, \binom{a^2}{b}, \dots, \binom{a^r}{b} =: S$ and corresponding digraph realizations G^1, G^2, \dots, G^r . We can use a slight relaxation of the proof from Theorem 6 for the case of loop-digraph sequences. (We only have to delete the demands $j \neq k, j' \neq k$ and $j \neq l$.) We get a generalization of Theorem 1 by Gale and Ryser.

Theorem 7. *Let $S' := \binom{a'}{b}$ be a loop-digraph sequence and $S := \binom{a}{b}$ be a sequence such that $a \prec a'$ and $a_1 \geq \dots \geq a_n$. Then there exist loop-digraph sequences $S' := \binom{a^1}{b}, \dots, \binom{a^r}{b} = S$ such that a^i yields a^{i+1} by a unit transfer, i.e. $a^{i+1} \prec a^i$. In particular, S is a loop-digraph sequence.*

Remark 2. *In Theorem 7 it is not necessary to sort loop-digraph sequence S if $t_{i,j}(a') = a$, i.e. $r = 2$. The reason is that this sorting is only necessary for Muirhead's Theorem 4. The construction of a loop-digraph does not need this sorting.*

We introduce a special notion for sequences $S' := \binom{a^1}{b}, \dots, \binom{a^r}{b} = S$ of (loop)-digraph sequences as they appear in Theorems 6 and 7.

Definition 2.2 (transfer path). *Sequences $S' := \binom{a^1}{b}, \dots, S^i, \dots, \binom{a^r}{b} = S$ of (loop)-digraph sequences S^i with $a^{i+1} \prec a^i$ and $t_{k_i, l_i}(a^i) = a^{i+1}$ where $k_i, l_i \in \{1, \dots, n\}$ are called transfer paths. We denote the value $(r - 1)$ as the length of a transfer path.*

Note, that the last three theorems give a proof for the existence of at least one transfer path between two (loop)-digraph sequences where one majorizes the other one. Clearly, there often exist several different such paths with different lengths. We give an example.

Example 2.1. *We consider the two loop-digraph sequences $S := \binom{2}{1}, \binom{2}{1}, \binom{2}{2}, \binom{0}{2}$ and $S' := \binom{4}{1}, \binom{2}{1}, \binom{0}{2}, \binom{0}{2}$. Then we find the following transfer paths.*

1. $S' = \binom{4}{1}, \binom{2}{1}, \binom{0}{2}, \binom{0}{2}, S^2 = \binom{3}{1}, \binom{3}{1}, \binom{0}{2}, \binom{0}{2}, S^3 = \binom{3}{1}, \binom{2}{1}, \binom{1}{2}, \binom{0}{2}, S^4 = \binom{2}{1}, \binom{2}{1}, \binom{2}{2}, \binom{0}{2}$
2. $S' = \binom{4}{1}, \binom{2}{1}, \binom{0}{2}, \binom{0}{2}, S^2 = \binom{3}{1}, \binom{2}{1}, \binom{1}{2}, \binom{0}{2}, S^3 = \binom{2}{1}, \binom{2}{1}, \binom{2}{2}, \binom{0}{2}$
3. $S' = \binom{4}{1}, \binom{2}{1}, \binom{0}{2}, \binom{0}{2}, S^2 = \binom{4}{1}, \binom{1}{1}, \binom{1}{2}, \binom{0}{2}, S^3 = \binom{3}{1}, \binom{2}{1}, \binom{1}{2}, \binom{0}{2}, S^4 = \binom{2}{1}, \binom{2}{1}, \binom{2}{2}, \binom{0}{2}$
4. $S' = \binom{4}{1}, \binom{2}{1}, \binom{0}{2}, \binom{0}{2}, S^2 = \binom{3}{1}, \binom{3}{1}, \binom{0}{2}, \binom{0}{2}, S^3 = \binom{2}{1}, \binom{3}{1}, \binom{1}{2}, \binom{0}{2}, S^4 = \binom{2}{1}, \binom{2}{1}, \binom{2}{2}, \binom{0}{2}$

The second transfer path is the one constructed in the proof of Theorem 7.

Note, that it is possible that the first components a^i are not decreasingly sorted (see our fourth transfer path). This is indeed important, because several digraph sequences do only possess threshold digraph sequences which are not decreasingly sorted. For that reason it was necessary to change the proof of Muirhead in Theorem 4.

3. The Connection between Majorization and the Number of Realizations

The main result of this section is to see that the number of (loop)-digraph realizations for a given (loop)-digraph sequence is smaller than the number of (loop)-digraph realizations for each majorized sequence. Indeed we prove that the number of (loop)-digraph realizations increases for each loop-digraph sequence in a transfer path. Since, each pair of sequences possesses at least one transfer path as we showed in the last section our claim can be proven by this approach. We want to give a short intuition for this idea. A

threshold (loop)-digraph sequence can only majorize other (loop)-digraph sequences and has only one unique realization. We asked if this property can be extended: Do sequences ‘near by’ threshold sequences possess only few realizations in contrast to sequences with a ‘large distance’ to a threshold sequence. This conjecture is true in a certain sense and can give for each sequence a ‘feeling’ of the number of realizations. Following this result, we can observe that *regular sequences* ($a_i = b_i = d$ for all $i \in \mathbb{N}_n$) possess the largest number of realizations in the set of all sequences with n tuples and fixed $m := \sum_{i=1}^n a_i$. The reason is that regular sequences can only be majorized by other sequences. Note, that regular sequences do only exist in the case $m \bmod n = 0$. It turns out, that (loop)-digraph sequences with a minimum sum of squared indegrees and a minimum sum of squared outdegrees possess the maximum number of realizations in the set of all sequences with fixed n and m . We call such (loop)-digraph sequences *minsquare sequences* or more general *minconvex sequences*. To see the connection between sum of squares (convex functions) and majorization consider the famous result of Polya, Littlewood and Hardy [HLP29].

Theorem 8 (Hardy, Polya, Littlewood 1929). *[HLP29] Let $g : \mathbb{R} \mapsto \mathbb{R}$ be an arbitrary convex function and $a, a' \in \mathbb{R}^n$. We have*

$$\sum_{i=1}^n g(a_i) \leq \sum_{i=1}^n g(a'_i)$$

if and only if $a \prec a'$.

There is a nice classic connection which is worth to be mentioned here. Let us define the following set DS with

$$DS := \{s = (a_1, \dots, a_n) \mid s \text{ is a graph sequence.}\}$$

Then the *polytope of graph sequences* is defined as the convex hull D_n of DS . Koren [Kor73] proved the following theorem.

Theorem 9 (Koren 1973). *A graph sequence is an extreme point of D_n if and only if it is a threshold sequence.*

The following ideas are folklore but we want to point out the interesting connection between majorization and threshold sequences. Hence, each graph sequence $s := (a_1, \dots, a_n)$ can be constructed by a convex combination of threshold sequences $x_i := (x_{i1}, \dots, x_{in})$, i.e. $s = \sum_{i=1}^k \lambda_i x_i$ where $\sum_{i=1}^k \lambda_i = 1$ ($\lambda_i > 0$). For an arbitrary convex function $g : \mathbb{R} \mapsto \mathbb{R}$ —using the inequality of Jensen—we get

$$\sum_{j=1}^n g(a_j) = \sum_{j=1}^n g\left(\sum_{i=1}^k \lambda_i x_{ij}\right) \leq \sum_{j=1}^n \sum_{i=1}^k \lambda_i g(x_{ij}) = \sum_{i=1}^k \lambda_i \sum_{j=1}^n g(x_{ij}).$$

We can conclude that there exists at least one x_i with $\sum_{j=1}^n g(x_{ij}) \geq \sum_{j=1}^n g(a_j)$. Assume this is not the case. Then we have

$$\sum_{i=1}^k \lambda_i \sum_{j=1}^n g(x_{ij}) < \sum_{i=1}^k \lambda_i \sum_{j=1}^n g(a_j) = \sum_{j=1}^n g(a_j)$$

in contradiction to our above inequality. By Theorem 8 there exists an x_i with $s \prec x_i$. Hence, the connection of majorization and threshold sequences follows from the property that threshold sequences are extreme points of the polytope of graph sequences. It is very natural to ask for the number of realizations of graph sequences that are not threshold sequences. Clearly, the number is larger than one. But a very intuitive idea for us was to ask for a more regular connection. We start our discussion with loop-digraph sequences, because the proofs are simpler than in the case of digraph sequences and graph sequences. Nevertheless, we try to develop our proofs in an analogous way for all cases.

3.1. The number of loop-digraph realizations and majorization

In a first step, we define the set $R(S)$ of all loop-digraph realizations for a given loop-digraph sequence $S = \binom{a}{b}$ and denote by $N(S) := |R(S)|$ the number of loop-digraph realizations of S . We want to give a connection between (i, j) -transfers on a loop-digraph sequence S' , i.e. $t_{i,j}(a') = a$ and a corresponding operation on a loop-digraph realization G' of S' leading to a loop-digraph realization G of S . Let us now consider the adjacency matrix A' for an arbitrary loop-digraph realization of sequence S' . Then a'_i, a'_j are the sums of the i th or j th columns, respectively. On the other hand, we find $a'_i \geq a'_j + 2$. Hence, it is possible to shift $|a'_i - a'_j| \geq 2$ entries ‘one’ from the i th column in the j th column. So, we can construct $|a'_i - a'_j| \geq 2$ different loop-digraph realizations of S from the one loop-digraph realization G' of S' with such shifts. More formally, we define:

Definition 3.1. We call an operation on the adjacency matrix $A' = (A'_{kl})_{k,l \in \{1, \dots, n\}}$ of loop-digraph realization $G' \in R(S')$, which switches the entries of $A'_{ki} = 1$ and $A'_{kj} = 0$ for one k , an (i, j) -shift on G' . We denote the subset of loop-digraph realizations from S which are constructed by (i, j) -shifts from a loop-digraph realization G' by $Shift_{ij}(G', S)$.

Let us now consider two arbitrary loop-digraph realizations of S' , namely G'_1 and G'_2 . Then it can happen that $Shift_{i,j}(G'_1, S) \cap Shift_{i,j}(G'_2, S) \neq \emptyset$. We give an example.

Example 3.1. We consider the adjacency matrices A'_1 and A'_2 of the two loop-digraph realizations G'_1 and G'_2 for loop-digraph sequence $S' := \binom{4}{1}, \binom{2}{1}, \binom{0}{1}, \binom{0}{1}, \binom{0}{1}, \binom{0}{1}$ and apply on each of them all 4 possible $(1, 2)$ -shifts. These shifts result in loop-digraph sequence $S = \binom{3}{1}, \binom{3}{1}, \binom{0}{1}, \binom{0}{1}, \binom{0}{1}, \binom{0}{1}$. Hence, $(3, 3, 0, 0, 0, 0) = t_{1,2}(4, 2, 0, 0, 0, 0)$. For

$$A'_1 := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A'_2 := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ we get}$$

$$Shift_{1,2}(G'_1, S) \cap Shift_{1,2}(G'_2, S) = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}. \text{ More generally}$$

there are $\binom{6}{4}$ possible loop-digraph realizations for S' and $\binom{6}{3}$ loop-digraph realizations for S .

Carefully comparing the two adjacency matrices we observe that their symmetrical difference contains exactly four arcs forming a directed alternating cycle, i.e. all four arcs alternate in their direction and in their appearance in G'_1 and G'_2 . Especially, *loop – arcs* are possible. Moreover, all of these arcs correspond to the first or second column of the matrices – the same columns as in the $(1, 2)$ – *shift*. In the following proposition we see that (i, j) -shifts on two loop-digraph realizations can lead to identical realizations if and only if the symmetric difference of their arc set is an alternating 4-cycle and contains arcs which can be shifted.

Proposition 1. *Let $S := \begin{pmatrix} a \\ b \end{pmatrix}$ and $S' := \begin{pmatrix} a' \\ b \end{pmatrix}$ be two different loop-digraph sequences with $t_{i,j}(a') = a$. Furthermore, we assume that G'_1 and G'_2 with $G'_1 \neq G'_2$ are loop-digraph realizations of S' . $\text{Shift}(G'_1, S) \cap \text{Shift}(G'_2, S) \neq \emptyset$ if and only if $A(G'_1) \Delta A(G'_2) = \{(k, i), (k', j), (k, j), (k', i)\}$ with $(k, i), (k', j) \in A(G'_1) \setminus A(G'_2)$ and $(k, j), (k', i) \in A(G'_2) \setminus A(G'_1)$ where $k \neq k'$.*

Proof. First we consider an arbitrary loop-digraph realization G with $G \in \text{Shift}(G'_1, S) \cap \text{Shift}(G'_2, S) \subset R(S)$. Then there exist two different (i, j) -shifts – one in G'_1 and one in G'_2 , say a shift changing $(k', i) \in A(G'_2)$ to $(k', j) \notin A(G'_2)$ and $(k, i) \in A(G'_1)$ to $(k, j) \notin A(G'_1)$. Since both (i, j) -shifts lead to the loop-digraph realization G , we can conclude $(k', i) \notin A(G'_1)$, $(k, i) \notin A(G'_2)$, $(k', j) \in A(G'_1)$ and $(k, j) \in A(G'_2)$. More differences between G'_1 and G'_2 cannot exist.

The converse implication holds trivially. For the two loop-digraph realizations G'_1 and G'_2 we define the (i, j) -shifts $(k', i) \in A(G'_1)$ to $(k', j) \notin A(G'_1)$, and $(k, i) \in A(G'_2)$ to $(k, j) \notin A(G'_2)$. Clearly, we get the same new realization G . \square

We call two loop-digraph realizations G'_1, G'_2 with a symmetric difference like in Proposition 1 (i, j) -adjacent. We define the subset $M(i, j) \subseteq R(S')$ of all loop-digraph realizations of S' as

$$M(i, j) := \{G' \in R(S') \mid \text{there exists a } G'_2 \in R(S') \text{ such that } G' \text{ and } G'_2 \text{ are } (i, j)\text{-adjacent.}\}$$

Then we get for two loop-digraph realizations $G'_1, G'_2 \in R(S') \setminus M(i, j)$ that $\text{Shift}_{i,j}(G'_1, S) \cap \text{Shift}_{i,j}(G'_2, S) = \emptyset$. Hence, the number of loop-digraph realizations which can be constructed by shifts from elements in $R(S') \setminus M$ is at least $|a'_i - a'_j| \cdot |R(S') \setminus M(i, j)| \geq 2 \cdot |R(S') \setminus M(i, j)|$.

Proposition 2. *Let $S := \begin{pmatrix} a \\ b \end{pmatrix}$ and $S' := \begin{pmatrix} a' \\ b \end{pmatrix}$ be two loop-digraph sequences with $t_{ij}(a') = a$. Furthermore, we assume that $G'_1, G'_2 \in R(S') \setminus M(i, j)$. Applying all possible (i, j) -shifts on all elements in $R(S') \setminus M(i, j)$ we get a subset of $R(S)$ which is at least twice the cardinality of $R(S') \setminus M(i, j)$. In particular, we get*

$$\left| \bigcup_{G' \in (R(S') \setminus M(i, j))} \text{Shift}_{i,j}(G', S) \right| \geq |a'_i - a'_j| \cdot |R(S') \setminus M(i, j)| \geq 2 \cdot |R(S') \setminus M(i, j)|.$$

Let us consider an extremal example for such a situation where we apply one after another $(i, i+1)$ -shifts for $i \in \{3, \dots, n-1\}$ on loop-digraph realizations which are not $(i, i+1)$ -adjacent. Clearly, this leads to exponentially many loop-digraph realizations.

Example 3.2. Let us consider the threshold loop-digraph sequence

$$S' := \binom{n-1}{0}, \binom{n-2}{1}, \binom{n-3}{2}, \dots, \binom{2}{n-3}, \binom{1}{n-2}, \binom{0}{n-1}$$

possessing exactly one loop-digraph realization. The Ferrers matrix of S' only possesses entries 0 above and on its main diagonal. Below this diagonal its entries are 1.

First we apply an $(1, 3)$ -shift and get two loop-digraph realizations of sequence S^2 . Then we apply one after another a $(3, 4)$ -shift, $(4, 5)$ -shift, \dots , $(n-1, n)$ -shift and get realizations for sequences $S^3, S^4, \dots, S^{n-1} =: S$. We get $S = \binom{n-2}{0}, \binom{n-2}{1}, \dots, \binom{1}{n-2}, \binom{1}{n-1}$. In each step we have two possibilities for a shift. On the other hand, there cannot be $(i, i+1)$ -adjacent realizations in $M(i, i+1) \subset R(S^{i-1})$, because realizations of $M(i, i+1) \subset R(S^{i-1})$ can only differ in the first i columns but not in the $(i+1)$ th column. Hence, $N(S) \geq 2^{n-2}$.

Note, that we have for sequence S and S' in our example $t_{1,n}(a') = a$. That is, only one shift would have been sufficient to achieve loop-digraph sequence S . In this case we only have $n-1$ possible $(1, n)$ -shifts and so our estimation for a lower bound of $N(S)$ is $n-1$. Hence, the kind of transfer paths plays an important role for the estimation of the possible lower bound for the number of realizations. In particular, the situation of the existence of two different (i, j) -adjacent loop-digraph realizations of a sequence S^i on a transfer path S', S^2, \dots, S^k (S' is threshold sequence) can only appear if there were (i, l) -shifts and (k, j) -shifts on transfer subpath S', \dots, S^{i-1} . Hence, we can conclude the following result.

Corollary 1. Let $S := \binom{a}{b}$ be a loop-digraph sequence and $S' := \binom{a'}{b'}$ its threshold loop-digraph sequence. If there exists a transfer path S', \dots, S of length k such that we have for each pair of transfers $t_{i,j}$ and $t_{i',j'}$, $i \neq i'$ and $j \neq j'$, then we have $N(S) \geq 2^k$.

Note, that it is possible that we have $j = i'$ or $i = j'$ as in Example 3.2. In the next steps, we need a combinatorial insight for binomial coefficients.

Proposition 3. Let $j' := i' - l$ where $i', j', l \in \mathbb{N}$. For $l \geq 1$ we have $\binom{2i'-l}{i'-l+1} \geq \binom{2i'-l}{i'}$. For $l \geq 2$ we have $\binom{2i'-l}{i'-l+1} > \binom{2i'-l}{i'}$.

Proof. We consider Pascal's triangle in a row $2i' - l$. For an even $2i' - l$, we find the maximum binomial coefficient $\binom{2i'-l}{i'-\frac{l}{2}}$. In this case l must be even and therefore $l \geq 2$. Clearly, the binomial coefficient decreases symmetrically starting on the maximum middle binomial coefficient in the directions of both borders of Pascal's triangle. Since, $|i' - (i' - \frac{l}{2})| = \frac{l}{2}$ and $|i' - \frac{l}{2} - (i' - l + 1)| = \frac{l}{2} - 1$, binomial coefficient $\binom{2i'-l}{i'-l+1}$ is nearer to the maximum binomial coefficient than $\binom{2i'-l}{i'}$. Hence, $\binom{2i'-l}{i'-l+1} > \binom{2i'-l}{i'}$. For an odd $2i' - l$, we find the two maximum binomial coefficients $\binom{2i'-l}{i'-\frac{l}{2}(l+1)}$ and $\binom{2i'-l}{i'-\frac{l}{2}(l-1)}$ in row $2i' - l$ of Pascal's triangle. Again, the binomial coefficients decrease symmetrically starting on the two maximum middle binomial coefficients in the directions of both borders of Pascal's triangle. Since we have for $l \geq 3$,

1. $|i' - (i' - \frac{l}{2}(l+1))| = \frac{l}{2}(l+1)$,
2. $|i' - l + 1 - (i' - \frac{l}{2}(l+1))| = \frac{l}{2}(l-3)$

3. $|i' - (i' - \frac{1}{2}(l-1))| = \frac{1}{2}(l-1)$,
4. $|i' - l + 1 - (i' - \frac{1}{2}(l-1))| = \frac{1}{2}(l-1)$ and

$\frac{1}{2}(l-3) < \frac{1}{2}(l-1)$ we get that $\binom{2i'-l}{i'-l+1}$ is nearer to the right maximum binomial coefficient than $\binom{2i'-l}{i'}$ to the left maximum binomial coefficient in Pascal's triangle. Hence, we have $\binom{2i'-l}{i'-l+1} > \binom{2i'-l}{i'}$ for $l \geq 2$. Let us now consider the case $l = 1$. We find that $\binom{2i'-l}{i'-l+1} = \binom{2i'-l}{i'}$. Hence, we have $\binom{2i'-l}{i'-l+1} \geq \binom{2i'-l}{i'}$ for $l \geq 1$. \square

Let us now consider all loop-digraph realizations which are constructed by (i, j) -shifts on elements in $M(i, j)$. Then we find the following result.

Proposition 4. *Let $S := \binom{a}{b}$ and $S' := \binom{a'}{b}$ be two different loop-digraph sequences with $t_{i,j}(a') = a$ and $M(i, j) \neq \emptyset$. Applying all possible (i, j) -shifts on all elements in $M(i, j) \subset R(S')$ we get a subset of loop-digraph realizations $R(S)$ which has a larger cardinality than $M(i, j)$, i.e. $|\bigcup_{G' \in M(i, j)} \text{Shift}(G', S)| > |M(i, j)|$.*

Proof. Two (i, j) -adjacent loop-digraph realizations in $M(i, j)$ do only differ in the i th and j th columns. There are at least $|a'_i - a'_j| \geq 2$ and at most a'_i possible (i, j) -shifts in each realization. More precisely, there exist i' (i, j) -shifts in each such realization with $2 \leq i' \leq a'_i$. Hence, we can conclude for an adjacency matrix of such a realization that there exist $i' \leq a'_i$ rows with entry "one" in column i whereas the entries in the j th columns of these rows are "zero". Consider a schematic picture where the permutation of the indices has been ignored. That means, the rows have been permuted to form four different blocks.

$A' :=$

$$\begin{array}{c}
 1 \\
 \vdots \\
 i' \\
 \vdots \\
 i' + j' \\
 \vdots \\
 n
 \end{array}
 \begin{pmatrix}
 1 & \dots & i & \dots & j & \dots & n \\
 \boxed{1} & & & & \boxed{0} & & \\
 \boxed{1} & & & & \boxed{0} & & \\
 \vdots & & & & \vdots & & \\
 \boxed{1} & & & & \boxed{0} & & \\
 \vdots & & & & \vdots & & \\
 \boxed{0} & & & & \boxed{1} & & \\
 \vdots & & & & \vdots & & \\
 \boxed{0} & & & & \boxed{1} & & \\
 \vdots & & & & \vdots & & \\
 \boxed{1} & & & & \boxed{1} & & \\
 \vdots & & & & \vdots & & \\
 \boxed{1} & & & & \boxed{1} & & \\
 \vdots & & & & \vdots & & \\
 \boxed{0} & & & & \boxed{0} & & \\
 \vdots & & & & \vdots & & \\
 \boxed{0} & & & & \boxed{0} & &
 \end{pmatrix}
 \begin{array}{c}
 \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} i' \\
 \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} j'
 \end{array}$$

a'_i

a'_j

Since $a'_i \geq a'_j + 2$, we find in the i th column $j' = i' - l$ rows with entries "zero" ($l \geq 2$) whereas the entries of these rows in the j th column are "one". There are different possibilities for i' and j' , but for two (i, j) -adjacent elements in $M(i, j)$ these values are fixed with Proposition 1. We have additionally $j' \geq 1$ and $i' \geq 3$. Otherwise there is no possibility for finding an (i, j) -adjacency for two realizations in $M(i, j)$. Hence, for each

fixed pair i', j' can either exist $\binom{i'+j'}{i'} = \binom{2i'-l}{i'}$ loop-digraph realizations of S' or no loop-digraph realization of S' in $M(i, j)$. This is true, because the entries in the corresponding $(i' + j')$ rows in columns i and j can be permuted and maintain the row sums. Hence, we can divide the set $M(i, j)$ in disjoint subsets $m(i', j')$ for each fixed pair (i', j') . That is

$$M(i, j) = \bigcup_{\substack{3 \leq i' \leq a'_i \\ 1 \leq j' \leq i' - l}} m(i', j').$$

It is sufficient to consider one arbitrary $m(i', j')$ which is not empty. Now, we can apply all possible (i, j) -shifts on each realization of $m(i', j')$. We get exactly $\binom{i'+j'}{j'+1} = \binom{2i'-l}{i'-l+1}$ different loop-digraph realizations of S . Since, $\binom{2i'-l}{i'-l+1} > \binom{2i'-l}{i'}$ for all non-empty $m(i', j')$ with $l \geq 2$ (Proposition 3) our proof is done. \square

We put the parts of all propositions together and get Theorem 10.

Theorem 10. *Let $S := \binom{a}{b}$ and $S' := \binom{a'}{b}$ be two different sequences with $t_{i,j}(a') = a$. Then it follows $N(S) \geq N(S')$. If S' is a loop-digraph sequence, then we have $N(S) > N(S')$.*

Proof. If S' is no loop-digraph sequence, then $N(S') = 0$ and the inequality holds trivially. So let us consider the case that S' is a loop-digraph sequence. Then S is also a loop-digraph sequence with Remark 2. A loop-digraph realization of S' is either in $M(i, j)$ or in $R(S') \setminus M(i, j)$. Now we apply for all realizations of S' all possible (i, j) -shifts. Then we get

$$\bigcup_{G' \in R(S')} Shift_{i,j}(G', S) \subset R(S).$$

(Note, that not all elements in $R(S)$ must be achieved by such shifts.) We apply Propositions 2 and 4 and get

$$\begin{aligned} N(S) &\geq \left| \bigcup_{G' \in R(S')} Shift_{i,j}(G', S) \right| \\ &= \left| \left(\bigcup_{G' \in R(S') \setminus M(i,j)} Shift_{i,j}(G', S) \right) \right| + \left| \left(\bigcup_{G' \in M(i,j)} Shift_{i,j}(G', S) \right) \right| \\ &\geq 2|R(S') \setminus M(i, j)| + |M(i, j)| \\ &> N(S') \end{aligned}$$

\square

If we now consider a transfer path between two loop-digraph sequences S and S' where S' majorizes S , we can easily conclude by Theorem 10 the following general result with respect to majorization.

Corollary 2. *Let $S := \binom{a}{b}$ with $a_1 \geq \dots \geq a_n$ and $S' := \binom{a'}{b}$ be two different loop-digraph sequences with $a \prec a'$. Then $N(S) > N(S')$.*

Proof. There exists at least one transfer path $S' := \binom{a^1}{b}, \dots, \binom{a^r}{b} =: S$ with $a^{i+1} \prec a^i$ by Theorem 7. We show by induction with respect to r the correctness of the claim. For $r = 2$ we apply Theorem 10 and get $N(S') < N(S)$. We consider the transfer path S^2, \dots, S . With our induction hypotheses we can conclude $N(S^2) < N(S)$. For S' and S^2 we apply again Theorem 10. This yields $N(S') < N(S^2) < N(S)$. \square

In Example 2.1 the loop-digraph sequence $S := \binom{2}{2}, \binom{2}{2}, \binom{1}{1}, \binom{1}{1}$ possesses the largest number of loop-digraph realizations in the set of all sequences with 4 tuples and $m = 6$. It is not the only loop-digraph sequence with this property. If we fix the a_i in S and permute the b_i we get loop-digraph sequences which all possess the same number of realizations. This can easily be seen if we consider an equivalent formulation of the loop-digraph realization problem, namely the bipartite realization problem. In each bipartite realization of sequence S we have only to permute the indices of the vertices v_i in the second independent vertex set corresponding to the current permutation of the b_i . Hence, the number of bipartite realizations is identical for each such permutation.

Proposition 5. *Let $S := \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n}$ be a loop-digraph sequence with $a_1 \geq \dots \geq a_n$ and $S_\tau := \binom{a_1}{b_{\tau(1)}}, \dots, \binom{a_n}{b_{\tau(n)}}$ a sequence where the second component of S was permuted by permutation $\tau : \mathbb{N}_n \mapsto \mathbb{N}_n$. Then S_τ is a loop-digraph sequence and the number of loop-digraph realizations of both is identical.*

Proof. We define a permutation matrix $P_\tau := (P_{i\tau(j)})_{i,j \in \mathbb{N}_n}$ with $P_{i\tau(j)} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

We consider for each loop-digraph realization of S or S_τ the corresponding adjacency matrix A or A_τ , respectively. Furthermore, we define a function $b : R(S_\tau) \mapsto R(S)$ with $b(A_\tau) := P_\tau A_\tau$. Since, b permutes the rows of A_τ (and therefore the ‘outdegrees’) we get the adjacency matrix of a loop-digraph realization of S . Since b is bijective, the number of realizations of S and S_τ is identical. \square

For the digraph realization problem this statement is not true. We consider the details in the next subsections. In a last step of this subsection we consider a special type of sequences. First we define an integer sequence $\alpha := (\alpha_1, \dots, \alpha_n)$ for a constant $m \in \mathbb{N}$ with $m \leq n^2$ by

$$\alpha_i := \begin{cases} m \operatorname{div}(n) + 1 & \text{for } i \in \{1, \dots, m \bmod (n)\} \\ m \operatorname{div}(n) & \text{for } i \in \{m \bmod (n) + 1, \dots, n\} \end{cases}$$

Clearly, $\sum_{i=1}^n \alpha_i = m$. If n is a factor of m , then we have $\alpha_i = \frac{m}{n}$ for all $i \in \mathbb{N}_n$.

Definition 3.2. *Let $\tau : |V| \mapsto |V|$ be an arbitrary permutation and $\alpha_\tau = (\alpha_{\tau(1)}, \dots, \alpha_{\tau(n)})$ a permutation of integer sequence α . We call a sequence*

$$S_{Min}^\tau := \begin{pmatrix} \alpha \\ \alpha_\tau \end{pmatrix}$$

minconvex sequence. For the identity permutation τ , we denote the minconvex sequence by S_{Min} .

In our next theorem we show that each decreasingly sorted integer sequence $a := (a_1, \dots, a_n)$ with $\sum_{i=1}^n a_i = m$ and $a_1 \geq \dots \geq a_n$ majorizes α . With Theorem 8 by Polya, Hardy and Littlewood this implies that $\sum_{i=1}^n g(a_i) \geq \sum_{i=1}^n g(\alpha_i)$ for all integer sequences a where $g : \mathbb{Z} \mapsto \mathbb{Z}$ denotes an arbitrary convex funktion. This is the intuition behind the notion minconvex sequence.

Theorem 11. *Let $a := (a_1, \dots, a_n)$ be a decreasingly sorted integer sequence with $\sum_{i=1}^n a_i = m$. Then we have $\alpha \prec a$.*

Proof. Assume there exists a $k < n$ with $\sum_{i=1}^k a_i < \sum_{i=1}^k \alpha_i$. Let k_0 the smallest of such k s. Since $\sum_{i=1}^{k_0-1} a_i \geq \sum_{i=1}^{k_0-1} \alpha_i$ for $k_0 > 1$ it follows $\alpha_{k_0} > a_{k_0} \geq a_{k_0+1} \geq \dots \geq a_n$. Then we get $\sum_{i=1}^n a_i < m$ in contradiction to our assumption. \square

Now, we are able to prove a general result for minconvex sequences.

Corollary 3. *Let $S := \binom{a}{b}$ be an arbitrary loop-digraph sequence with $\sum_{i=1}^n a_i = m$, $a_1 \geq \dots \geq a_n$ which is no minconvex sequence. Then we find for an arbitrary minconvex sequence S_{Min}^τ that $N(S_{Min}^\tau) > N(S)$.*

Proof. We define the following sequences.

1. $S' := \binom{\alpha}{b}$ and $S'_\tau := \binom{\alpha_\tau}{b_\tau}$ with decreasingly sorted $b_{\tau(i)}$.
2. $S'' := \binom{\alpha_\tau}{\alpha}$.

It is easy to see that $\alpha \prec a$ and $\alpha \prec b_\tau$ (Theorem 11). Hence, we find by Corollary 2 that $N(S) < N(S')$ and $N(S'_\tau) < N(S'')$ if we change the components a_τ and b_τ of S'_τ and a_τ and a in S'' . Clearly, $N(S') = N(S'_\tau)$ and $N(S'') = N(S_{Min}^\tau)$. It follows $N(S) < N(S_{Min}^\tau)$. With Proposition 5 our proof is done. \square

3.2. The number of digraph realizations and majorization

In this subsection we try to follow the proofs of the last subsection about loop-digraph realizations for the digraph realization problem. We start with an example showing that we cannot apply all approaches analogously. It turns out that we have to modify the main result in Corollary 2. In particular, it is necessary to sort a digraph sequence S in decreasing lexicographical order. For simplicity we take the notions of $R(S)$ and $N(S)$ of the last subsection and transfer it to the case of a digraph sequence S . Furthermore, we define (i, j) -shifts.

Definition 3.3. *We call an operation on the adjacency matrix $A' := (A'_{il})_{i,l \in \{1, \dots, n\}}$ of digraph realization $G' \in R(S')$, which switches the entries of $A'_{ki} = 1$ and $A'_{kj} = 0$ where $k \neq j$ an (i, j) -shift on G' . We denote the subset of digraph realizations of S which are constructed by (i, j) -shifts from a digraph realization G' by $Shift_{ij}(G', S)$.*

Let us start with an example to discuss some problems leading to the above mentioned restriction.

Example 3.3. *First we apply the unique $(3, 4)$ -shift on the adjacency matrix A' of threshold sequence $S' := \binom{2}{0}, \binom{2}{2}, \binom{2}{1}, \binom{0}{3} =: \binom{a'}{b}$ leading to the adjacency matrix A of*

another threshold sequence $S = \binom{2}{0}, \binom{2}{2}, \binom{1}{1}, \binom{1}{3} =: \binom{a}{b}$, where $A' := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ and

$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. It follows $N(S') = N(S) = 1$. On the other hand we have $a \prec a'$.

Hence, a strict analogous result of Theorem 10 for digraph sequences is not possible.

The reason for this observation is that it is only possible to apply one unique (i, j) -shift in contrast to Proposition 2 where always at least two shifts are possible. On the other hand it turns out (see Theorem 13), that this result is true if digraph sequence S is decreasingly lexicographical sorted. However, we cannot transfer all ideas from the proofs of the loop-digraph realization problem. To see this, consider the following example.

Example 3.4. Consider the adjacency matrix A' of threshold sequence $S' := \binom{2}{1}, \binom{1}{1}, \binom{0}{1}$. We have $A' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. We apply the unique $(1, 3)$ -shift and get $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Note that the digraph sequence $S = \binom{1}{1}, \binom{1}{1}, \binom{1}{1}$ of A is lexicographically sorted. On the other hand there exists another different digraph realization G^* of S , i.e. $A^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Hence, we have $N(S') > N(S)$ but A^* cannot be achieved by a shift.

It is easy to transfer the concept of (i, j) -adjacent loop-digraph realizations $G'_1, G'_2 \in R(S')$ –with some little restrictions to avoid loops– to digraph realizations.

Proposition 6. Let $S := \binom{a}{b}$ and $S' := \binom{a'}{b'}$ be two different digraph sequences with $t_{ij}(a') = a$. Furthermore, we assume that G'_1 and G'_2 are digraph realizations of S' with $G'_1 \neq G'_2$. $\text{Shift}(G'_1, S) \cap \text{Shift}(G'_2, S) \neq \emptyset$ if and only if $A(G'_1) \Delta A(G'_2) = \{(k, i), (k', j), (k, j), (k', i)\}$ with $(k, i), (k', j) \in A(G'_1) \setminus A(G'_2)$ and $(k, j), (k', i) \in A(G'_2) \setminus A(G'_1)$ where $k \notin \{k', i, j\}$ and $k' \notin \{k, i, j\}$.

The proof can be done like in Proposition 1. We call two digraph realizations G'_1, G'_2 with a symmetric difference like in Proposition 6 (i, j) -adjacent. Furthermore, we also use the notion of $M(i, j)$ as the set of all digraph realizations in $R(S')$ which possess at least one (i, j) -adjacent digraph realization. We prove the claim of this subsection in two steps. First we transfer the proofs of the last chapter showing the results in a restricted case. This approach is analogously to the the loop-digraph realization problem. In a second step we prove the results for the stronger case. Here we have to change some approaches. Lastly, we consider minconvex sequences for digraph sequences. In this case we again have to modify our results with respect to loop-digraph sequences. It turns out, that minconvex sequences $S_{Min} = \binom{\alpha}{\alpha_\tau}$ possess the largest number of digraph realizations if τ sorts the α_i of α in increasing order, i.e. $\alpha_{\tau(1)} \leq \dots \leq \alpha_{\tau(n)}$.

Proposition 7. Let $S := \binom{a}{b}$ and $S' := \binom{a'}{b'}$ be two different digraph sequences with $t_{ij}(a') = a$. Furthermore, we assume that $G'_1, G'_2 \in R(S') \setminus M(i, j)$. Applying all possible

(i, j) -shifts on all elements in $R(S') \setminus M(i, j)$ we get a subset of $R(S)$ which is larger or equals the cardinality of $R(S') \setminus M(i, j)$. In particular,

$$|\bigcup_{G' \in (R(S') \setminus M)} \text{Shift}_{i,j}(G', S)| \geq (|a'_i - a'_j| - 1) \cdot |R(S') \setminus M(i, j)| \geq |R(S') \setminus M(i, j)|.$$

Proof. Consider an adjacency matrix $A' := (A'_{il})_{i,l \in \{1, \dots, n\}}$ of a digraph realization $S' \in R(S')$. With our assumption we have at least $|a'_i - a'_j|$ entries $A'_{ki} = 1$ with $A'_{kj} = 0$. Clearly, $k \neq i$ but it can happen that $k = j$. In this case there exist at least $(|a'_i - a'_j| - 1)$ (i, j) -shifts. With our definition of transfers we have $a'_i \geq a'_j + 2$. This proves our claim. \square

Proposition 8. Let $S := \begin{pmatrix} a \\ b \end{pmatrix}$ and $S' := \begin{pmatrix} a' \\ b' \end{pmatrix}$ be two different digraph sequences with $t_{ij}(a') = a$. Applying all possible (i, j) -shifts on all elements in $M(i, j) \subset R(S')$ we get a subset of digraph realizations $R(S)$ which is larger or equals the cardinality of $M(i, j)$, i.e. $|\bigcup_{G' \in M(i, j)} \text{Shift}(G', S)| \geq |M(i, j)|$.

Proof. Let $G' \in M(i, j)$. Then there exist four different cases for a ‘schematic’ adjacency matrix A' of G' resulting of the possible differences in the i th and j th columns. Let us consider such a schematic picture.

$A' =$

$$\begin{array}{c} 1 \\ \vdots \\ i' \\ \vdots \\ i' + j' \\ i \\ j \\ \vdots \\ n \end{array} \begin{pmatrix} 1 & \dots & i & \dots & j & \dots & n \\ \boxed{1} & & & & \boxed{0} & & \\ \boxed{1} & & & & \boxed{0} & & \\ \vdots & & & & \vdots & & \\ \boxed{1} & & & & \boxed{0} & & \\ \vdots & & & & \vdots & & \\ \boxed{0} & & & & \boxed{1} & & \\ \vdots & & & & \vdots & & \\ \boxed{0} & & & & \boxed{1} & & \\ \vdots & & & & \vdots & & \\ \boxed{0} & & & & \boxed{1} & & \\ \vdots & & & & \vdots & & \\ \boxed{0} & & & & \boxed{0} & & \\ \vdots & & & & \vdots & & \\ \boxed{0} & & & & \boxed{0} & & \end{pmatrix} \left. \begin{array}{l} \left. \begin{array}{l} \vdots \\ \vdots \end{array} \right\} i' \\ \left. \begin{array}{l} \vdots \\ \vdots \end{array} \right\} j' \end{array} \right\}$$

$\boxed{0}$
 a'_i

$\boxed{a'_{ij}}$
 a'_j

Note, that $j' \geq 1$, otherwise $G' \notin M(i, j)$. We distinguish between four different cases applying our assumption $a'_i \geq a'_j + 2$.

case 1: $a'_{ij} = 0$ and $a'_{ji} = 1$. Then we have with $j' := i' - l$ at least 2 possible (i, j) -shifts. It follows $l \geq 1$.

case 2: $a'_{ij} = 0$ and $a'_{ji} = 0$. Then we have with $j' := i' - l$ at least 3 possible (i, j) -shifts. It follows $l \geq 2$.

case 3: $a'_{ij} = 1$ and $a'_{ji} = 1$. Then we have with $j' := i' - l$ at least 3 possible (i, j) -shifts. It follows $l \geq 2$.

case 4: $a'_{ij} = 1$ and $a'_{ji} = 0$. Then we have with $j' := i' - l$ at least 4 possible (i, j) -shifts. It follows $l \geq 3$.

In all four cases there do exist $\binom{i'+j'}{i'} = \binom{2i'-l}{i'}$ digraph realizations or no digraph realization. The reason is that the entries of the $(i'+j')$ rows can be permuted maintaining the row sums. After all possible (i, j) -shifts in each non-empty case, we get $\binom{i'+j'}{j'+1} = \binom{2i'-l}{i'-l+1}$ digraph realizations for each case. Applying Proposition 3 we obtain our claim. Note, that equality can only appear in case 1. \square

Theorem 12. Let $S := \binom{a}{b}$ and $S' := \binom{a'}{b}$ be two different sequences with $t_{ij}(a') = a$. Then it follows $N(S) \geq N(S')$.

Proof. If S' is no digraph sequence, then $N(S') = 0$ and the inequality holds trivially. So let us consider the case that S' is a digraph sequence. Then S is also a digraph sequence with Remark 1. A digraph realization of S' is either in $M(i, j)$ or in $R(S') \setminus M(i, j)$. Now we apply for all these realizations of S' all possible (i, j) -shifts. Then we get

$$\bigcup_{G' \in R(S')} \text{Shift}_{i,j}(G', S) \subset R(S).$$

(Note, that not all elements in $R(S)$ must be achieved by such shifts.) We apply Propositions 7 and 8 and get

$$\begin{aligned} N(S) &\geq \left| \bigcup_{G' \in R(S')} \text{Shift}_{i,j}(G', S) \right| \\ &= \left| \left(\bigcup_{G' \in R(S') \setminus M(i,j)} \text{Shift}_{i,j}(G', S) \right) \cup \left(\bigcup_{G' \in M(i,j)} \text{Shift}_{i,j}(G', S) \right) \right| \\ &\geq |R(S') \setminus M(i, j)| + |M(i, j)| \\ &= N(S'). \end{aligned}$$

\square

Theorem 13. Let $S := \binom{a}{b}$ be a lexicographical decreasingly sorted sequence and $S' := \binom{a'}{b}$ a sequence such that $t_{ij}(a') = a$. Then it follows $N(S) \geq N(S')$. If S' is a digraph sequence, then we have $N(S) > N(S')$.

Proof. If S' is no digraph sequence, then $N(S') = 0$ and the inequality holds trivially. So let us consider the case that S' is a digraph sequence. Then S is also a digraph sequence with Remark 1. We distinguish between the cases, that $G' \in M(i, j)$ and $G' \in R(S') \setminus M(i, j)$. For $G' \in R(S') \setminus M(i, j)$ and $a'_i \geq a'_j + 3$ we always find at least two possible (i, j) -shifts for each such digraph realization. For $G' \in M(i, j)$ and $a'_i \geq a'_j + 3$ consider all four cases in the proof of Proposition 8. Clearly, we find there $l \geq 2$ in case 1. With Proposition 3 we can prove our claim. Let us now assume $a'_i := a'_j + 2$. Again

we consider a digraph realization $G' \in R(S') \setminus M(i, j)$. Then we find for an adjacency matrix A' one of the following four schematic scenarios.

$$A' := \begin{matrix} & 1 & \dots & i & \dots & j & \dots & n \\ \begin{matrix} 1 \\ \vdots \\ i' \\ i \\ j \\ \vdots \\ n \end{matrix} & \left(\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \end{array} \right) & \left. \begin{matrix} \begin{matrix} 1 \\ 1 \\ \vdots \\ 1 \end{matrix} \\ \begin{matrix} 0 \\ a'_{ji} \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{matrix} \right\} i'\end{matrix}$$

a'_i

a'_j

Since, $a'_i = a'_j + 2$ we get

case 1: $a'_{ij} = 0$ and $a'_{ji} = 1$. Then we have $i' = 1$ and so one possible (i, j) -shift.

case 2: $a'_{ij} = 0$ and $a'_{ji} = 0$. Then we have $i' = 2$ and so two possible (i, j) -shifts.

case 3: $a'_{ij} = 1$ and $a'_{ji} = 1$. Then we have $i' = 2$ and so two possible (i, j) -shifts.

case 4: $a'_{ij} = 1$ and $a'_{ji} = 0$. Then we have with $i' = 3$ at so at least three possible (i, j) -shifts.

If we find at least one digraph realization $G' \in M(i, j)$ for cases 2, 3, 4 —consider the proof of Proposition 7— or for cases 2, 3, 4 above when $G' \in R(S') \setminus M(i, j)$, we get $N(S) > N(S')$. So, let us assume, that there only exist digraph realizations in case 1 in the proof of Proposition 7 or case 1 above. Furthermore, with our additional assumption we have $a'_i = a'_j + 2$ and $l = 1$. Then arc (i, j) does not belong to any digraph realization of S' . Since, $a_i = a_j$ and S is lexicographical decreasingly sorted, we have $b_i \geq b_j$. Since, $a'_{ji} = 1$, there exists in an arbitrary digraph realization G' a $k \neq i, j$ with $a'_{ik} = 1$ and $a'_{jk} = 0$. On the other hand we have our unique (i, j) -shift and so for a $k' \neq i, j$, $a'_{k'i} = 1$ and $a'_{k'j} = 0$. Note, that we can have $k = k'$. After applying the (i, j) -shift in G' we get a digraph realization $G \in R(S)$ containing a directed 3-path $p := (k', j, i, k)$. We get a further digraph realization G^* in $R(S)$ if we change path p to $p^* := (k', i, j, k)$. This is possible, since arc (k', i) and (i, j) and (j, k) are not in G . On the other hand no (i, j) -shift can realize digraph G^* , because with our assumption no G' contains arc (i, j) and the shifts do not construct this arc. Hence, there exists at least one digraph realization more for S than for S' . \square

Corollary 4. Let $S := \binom{a}{b}$ be a lexicographical decreasingly sorted digraph sequence and $S' := \binom{a'}{b}$ be a digraph sequence with $a \prec a'$. Then $N(S) > N(S')$.

Proof. There exists at least one transfer path $S' := (a^1_b), \dots, (a^r_b) =: S$ with $a^{i+1} \prec a^i$ and $t_{k,l+1}(a^i) = a^{i+1}$ by Theorem 6. We show by induction with respect to r the correctness of the claim. For $r = 2$ we apply Theorem 13 and get $N(S') < N(S)$. Let us now assume $r > 2$. We consider the transfer path S^2, \dots, S . With our induction hypotheses we can conclude $N(S^2) < N(S)$. For S' and S^2 we apply Theorem 12. This yields $N(S') \leq N(S^2) < N(S)$. \square

Back to our Example 3.4 we find with Theorem 13 that $N(S') < N(S)$. Here the directed path is a directed cycle of length three. In Example 3.3 we have to sort digraph sequence S in lexicographical order. That is $S_\sigma = \binom{2}{2}, \binom{2}{0}, \binom{1}{3}, \binom{1}{1} =: \binom{a_\sigma}{b_\sigma}$. Hence,

the adjacency matrix is its Ferrers matrix $A := \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. If we sort S'_σ in an

identical way we get $S'_\sigma = \binom{2}{2}, \binom{2}{0}, \binom{0}{3}, \binom{2}{1} =: \binom{a'_\sigma}{b'_\sigma}$. Hence we have $a'_\sigma \prec a_\sigma$, but S'_σ is not lexicographically sorted. Hence, it is not possible to apply Theorem 13.

In the next step we want to show that digraph sequences which possess a so-called opposed ordering have at least as many realizations as a sequence with a permutation of the second component. This leads to the result that opposed minconvex sequences possess the largest number of digraph realizations under all digraph realizations with n tuples and fixed $m = \sum_{i=1}^n a_i$. We start with a definition of these sequences.

Definition 3.4. Let $S := \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n}$ be a digraph sequence with $a_1 \geq \dots \geq a_n$ and $b_1 \leq \dots \leq b_n$, then we denote S by opposed sequence.

Proposition 9. Let $S := \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n}$ be a digraph sequence with $a_1 \geq \dots \geq a_n$ and $S_\tau := \binom{a_1}{b_{\tau(1)}}, \dots, \binom{a_n}{b_{\tau(n)}}$ a sequence where the second component of S was permuted by transposition $\tau : \mathbb{N}_n \mapsto \mathbb{N}_n$ such that there exist $l < m$ with $b_l > b_m$ and $b_{\tau(l)} < b_{\tau(m)}$. Then S_τ is a digraph sequence and $N(S_\tau) \geq N(S)$.

Proof. We define a transposition matrix $P_\tau := (P_{i,j})_{i,j \in \mathbb{N}_n}$ with

$$P_{ij} := \begin{cases} 1 & \text{if } i = j \wedge i, j \neq l, m \\ 1 & \text{if } i = l, j = m \vee i = m, j = l \\ 0 & \text{otherwise} \end{cases}$$

We consider for each digraph realization of S or S_τ the corresponding adjacency matrix A or A_τ , respectively. First we give an instruction how to construct for each digraph realization $G \in R(S)$ one or two digraph realizations G_τ of $R(S_\tau)$. In some cases it can happen that two of such constructed digraph realizations are identical. Nevertheless, we prove that $N(S_\tau) \geq N(S)$. Furthermore, we define a function $b : R(S) \mapsto R(S_\tau)$ for four different cases with $b(A) := P_\tau A - F$ where $F := (F_{i,j})_{i,j \in \mathbb{N}_n}$ is a matrix which is defined in the following way.

case 1: For $A_{lm} = 0$ and $A_{ml} = 0$ we set $A_\tau := P_\tau A$ and $F := 0$.

case 2: For $A_{lm} = 1$ and $A_{ml} = 1$ we set $A_\tau := P_\tau A - F$ with $F_{ll} = F_{mm} = 1$, $F_{lm} = F_{ml} = -1$ and all other entries $F_{ij} = 0$.

case 3: For $A_{lm} = 1$ and $A_{ml} = 0$ we find a $k \neq l, m$ with $A_{kl} = 1$ and $A_{km} = 0$, because $a_l \geq a_m$. Since we get an identical sequence $S_\tau = S$ for the case $a_l = a_m$ (up to permutation), we can assume that $a_l > a_m$. Hence, there exists a further $k' \neq l, m, k$ with $A_{k'l} = 1$ and $A_{k'm} = 0$. We set $A_\tau^1 := P_\tau A - F^1$ with $F_{mm}^1 = F_{kl}^1 = 1$, $F_{km}^1 = F_{ml}^1 = -1$ and all other entries $F_{ij}^1 = 0$. Furthermore, we construct a further digraph realization $A_\tau^2 := P_\tau A - F^2$ with $F_{k'l}^2 = F_{mm}^2 = 1$, $F_{k',m}^2 = F_{m,l}^2 = -1$ and all other entries $F_{ij}^2 = 0$. Note, that all digraph realizations of S_τ in this case are pairwise distinct. Otherwise, we would find A^1 and A^2 with $P_\tau A^1 - F^1 = P_\tau A^2 - F^2$. This is equivalent to $A^1 - A^2 =: A^*$ with $A_{kl}^* = 1$. It follows $A_{kl}^1 = 1$ and $A_{kl}^2 = 0$ in contradiction to our assumption in this case that $A_{kl}^2 = 1$.

case 4: For $A_{lm} = 0$ and $A_{ml} = 1$ we find $k, k' \neq l, m$ with $k \neq k'$ and $A_{lk} = A_{lk'} = 1$ and $A_{mk} = A_{mk'} = 0$, because we have with our assumption $b_l > b_m$. We set $A_\tau^1 := P_\tau A - F^1$ with $F_{ll}^1 = F_{mk}^1 = 1$, $F_{lk}^1 = F_{ml}^1 = -1$ and all other entries $F_{ij}^1 = 0$. Furthermore, we construct a further digraph realization $A_\tau^2 := P_\tau A - F^2$ with $F_{ll}^2 = F_{mk'}^2 = 1$, $F_{lk'}^2 = F_{ml}^2 = -1$ and all other entries $F_{ij}^2 = 0$. Note, that digraph realizations of S_τ in this case are pairwise distinct. Otherwise, we would find A^1 and A^2 with $P_\tau A^1 - F^1 = P_\tau A^2 - F^2$. This is equivalent to $A^1 - A^2 =: A^*$ with $A_{mk}^* = 1$. It follows $A_{mk}^1 = 1$ and $A_{mk}^2 = 0$ in contradiction to our assumption in this case that $A_{mk}^1 = 0$.

Let us denote the digraph realizations of S_τ in case 1 to case 4 by $R_1(S_\tau), R_2(S_\tau), R_3(S_\tau)$ and $R_4(S_\tau)$ and the digraph realizations of S in these cases by $R_1(S), R_2(S), R_3(S), R_4(S)$. It is easy to see that $R_i(S_\tau) \cap R_j(S_\tau) = \emptyset$ for the following pairs of $(i, j) \in \{(1, 2), (2, 3), (1, 3), (1, 4), (2, 4)\}$. The reason is that we find by our assumption for a digraph of S_τ , G_τ or G_τ^1, G_τ^2 , respectively, that

case 1: $(l, m), (m, l) \notin A(G_\tau)$,

case 2: $(l, m), (m, l) \in A(G_\tau)$,

case 3: $(l, m) \notin A(G_\tau^i)$ and $(m, l) \in A(G_\tau^i)$ where $i \in \{1, 2\}$, and

case 4: $(l, m) \notin A(G_\tau^i)$ and $(m, l) \in A(G_\tau^i)$ where $i \in \{1, 2\}$.

Note, it can happen that $R_3(S_\tau) \cap R_4(S_\tau) \neq \emptyset$. Without loss of generality, we assume that $|R_3(S)| \geq |R_4(S)|$. Clearly, with our instruction of case 3, we construct in this case $2|R_3(S)|$ digraph realizations of S_τ . Hence, we find

$$\begin{aligned} N(S_\tau) &\geq |R_1(S_\tau) \cup R_2(S_\tau) \cup R_3(S_\tau) \cup R_4(S_\tau)| \\ &\geq |R_1(S)| + |R_2(S)| + 2|R_3(S)| \\ &\geq |R_1(S)| + |R_2(S)| + |R_3(S)| + |R_4(S)| \\ &= N(S). \end{aligned}$$

□

Since each S can be permuted by a sequence of transpositions in its opposed sequence S_σ we can apply the last Proposition at each step and get the following result.

Theorem 14. Let $S := \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n}$ be a digraph sequence with $a_1 \geq \dots \geq a_n$ and $S_\sigma := \binom{a_1}{b_{\sigma(1)}}, \dots, \binom{a_n}{b_{\sigma(n)}}$ its opposed sequence where the second component of S was permuted by a permutation $\sigma : \mathbb{N}_n \mapsto \mathbb{N}_n$. Then S_σ is a digraph sequence and $N(S_\sigma) \geq N(S)$.

Consider two simple examples. $S := \binom{2}{2}, \binom{2}{1}, \binom{1}{2}$ and its opposed sequence $S_\sigma := \binom{2}{1}, \binom{2}{2}, \binom{1}{2}$ are both threshold sequences and each possesses exactly one digraph realization. $S := \binom{2}{1}, \binom{1}{0}, \binom{0}{2}$ is not a digraph sequence but its opposed sequence $S_\sigma := \binom{2}{0}, \binom{1}{1}, \binom{0}{2}$ is a threshold sequence. Hence we have $N(S) < N(S_\sigma)$.

In a last step of this subsection we again consider minconvex sequences. First we repeat the definition of integer sequence $\alpha := (\alpha_1, \dots, \alpha_n)$ for a constant $m \in \mathbb{N}$ with $m \leq \binom{n}{2}$ by

$$\alpha_i := \begin{cases} m \operatorname{div}(n) + 1 & \text{for } i \in \{1, \dots, m \bmod (n)\} \\ m \operatorname{div}(n) & \text{for } i \in \{m \bmod (n) + 1, \dots, n\} \end{cases}$$

Definition 3.5. Let $\tau : |V| \mapsto |V|$ be an arbitrary permutation and $\alpha_\tau = (\alpha_{\tau(1)}, \dots, \alpha_{\tau(n)})$ a permutation of integer sequence α . We call a sequence

$$S_{Min}^\tau := \begin{pmatrix} \alpha \\ \alpha_\tau \end{pmatrix}$$

minconvex sequence.

Now, we are able to prove the general result for minconvex digraph sequences.

Corollary 5. Let $S := \binom{a}{b}$ be an arbitrary digraph sequence with $\sum_{i=1}^n a_i = m$, $a_1 \geq \dots \geq a_n$. Then we find for the opposed minconvex sequence S_{Min} that $N(S_{Min}) \geq N(S)$.

Proof. We define the following sequences.

1. $S' := \binom{\alpha}{b}$ and $S'_\tau := \binom{\alpha_\tau}{b_\tau}$ with decreasingly sorted $b_{\tau(i)}$.
2. $S'' := \binom{\alpha_\tau}{\alpha}$.

It is easy to see that $\alpha \prec a$ and $\alpha \prec b_\tau$ (Theorem 11). Hence, we find by an iterative application of Theorem 12 and Remark 1 that $N(S) \leq N(S')$ and $N(S'_\tau) \leq N(S'')$ if we change the components of S'_τ and S'' . Clearly, $N(S') = N(S'_\tau)$ and $N(S'') \leq N(S_{Min})$ with Theorem 14. It follows $N(S) \leq N(S_{Min})$. \square

3.3. The number of graph realizations and majorization

The connection between the number of graph realizations and majorization can easily be proved using the results of the last subsection. For that we consider instead the graph sequence $s := (a_1, \dots, a_n)$ digraph sequence $S := \binom{a_1}{a_1}, \dots, \binom{a_n}{a_n}$ and restrict the set of all digraph realizations to symmetric digraphs, i.e. for each $(v, w) \in A(G)$ also exists $(w, v) \in A(G)$. The set of all graph realizations we denote by $R(s)$ and we set $N(s) := |R(s)|$. The set of all symmetric digraph realizations of S we denote by $R(S)$ and set $N(S) := |R(S)|$. Clearly, we cannot use the notion of shifts of the last two subsections, because this would result in a non-symmetric digraph realization. On the other hand, we can simply distinguish between *horizontal* (i, j) -shifts and *vertical* (i, j) -shifts. Hence, horizontal (i, j) -shifts for an adjacency matrix A' of $G' \in R(s')$ are exactly

the (i, j) -shifts of Definition 3.3. Vertical (i, j) -shifts correspond to horizontal (i, j) -shifts in the sense that the symmetric entries have to be shifted in the corresponding columns. (Note, that with this definition loops will be excluded.) Clearly, if we always apply horizontal and vertical shifts simultaneously we again yield a symmetric realization and so a graph realization. As in the Proposition 6 two symmetric digraph realizations are (i, j) -adjacent if their symmetric difference can be found on the i th and j th columns of their adjacency matrices. Clearly, there exist the corresponding symmetric difference on their rows. We use the notion of (i, j) -adjacency in this new sense. We transfer the notion of $M(i, j)$ here to the set of all *symmetric digraph realizations* of S which possess at least one (i, j) -adjacent symmetric digraph realization in $R(S)$. It turns out, that the case of the graph realization problem is more simple than the case of the digraph realization problem. One reason is that each S is decreasingly lexicographical sorted if the s is decreasingly sorted. Another point is the absence of difficult cases as they appear in the proof of Theorem 13.

Theorem 15. *Let $s := (a_1, \dots, a_n)$ be a decreasingly sorted integer sequence and $s' := (a'_1, \dots, a'_n)$ an integer sequence such that $t_{ij}(s') = s$. Then it follows $N(s) \geq N(s')$. If s' is a graph sequence, then we have $N(s) > N(s')$.*

Proof. We consider sequences $S := \binom{a_1}{a_1}, \dots, \binom{a_n}{a_n}$ and $S' := \binom{a'_1}{a'_1}, \dots, \binom{a'_n}{a'_n}$. We distinguish for $G' \in R(S')$ between the two cases $G' \in M(i, j)$ and $G' \in R(S') \setminus M(i, j)$. Consider the following two possible schematic matrices for these cases.

case 1: $G' \in R(S') \setminus M(i, j)$. We consider a schematic picture for different i' . $A' :=$

$$\begin{array}{c}
 1 \\
 \vdots \\
 i' \\
 i \\
 j \\
 \vdots \\
 n
 \end{array}
 \left(
 \begin{array}{ccccccc}
 1 & \dots & i' & i & \dots & j & \dots & n \\
 \vdots & & & \boxed{\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array}} & & \boxed{\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array}} & & \\
 i' & & & \boxed{1} & & \boxed{0} & & \\
 i & 1 & \dots & 1 & & a'_{ij} & & \\
 j & 0 & \dots & 0 & & \boxed{0} & & \\
 \vdots & & & \boxed{\begin{array}{c} 1 \\ \vdots \\ 1 \end{array}} & & \boxed{\begin{array}{c} 1 \\ \vdots \\ 1 \end{array}} & & \\
 n & & & \boxed{\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}} & & \boxed{\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}} & &
 \end{array}
 \right)
 \left. \vphantom{\begin{array}{c} 1 \\ \vdots \\ i' \\ i \\ j \\ \vdots \\ n \end{array}} \right\} i'$$

Since, A' is symmetric we have the two possibilities a) $a'_{ij} = 0$ and $a'_{ji} = 0$ and b) $a'_{ij} = 1$ and $a'_{ji} = 1$. Hence, we have at least two possible vertical and two possible horizontal (i, j) -shifts. It follows that the number of symmetric digraph realizations of S is at least twice the cardinality of $R(S') \setminus M(i, j)$.

case 2: $G' \in M(i, j)$. Again we consider all possible scenarios for pairs (i', j') see the proof of Proposition 8. We show a schematic picture.

$$A' := \begin{pmatrix} 1 & \dots & i' & \dots & & & & i & j & \dots & n \\ 1 & & & & & & & \boxed{1} & \boxed{0} & & \\ \vdots & & & & & & & \boxed{1} & \boxed{0} & & \\ i' & & & & & & & \boxed{\vdots} & \boxed{\vdots} & & \\ \vdots & & & & & & & \boxed{1} & \boxed{0} & & \\ i' + j' & & & & & & & \boxed{0} & \boxed{1} & & \\ i & 1 & \dots & 1 & 0 & \dots & 0 & \boxed{0} & \boxed{a'_{ij}} & & \\ j & 0 & \dots & 0 & 1 & \dots & 1 & \boxed{a'_{ji}} & \boxed{0} & & \\ \vdots & & & & & & & \boxed{1} & \boxed{1} & & \\ & & & & & & & \boxed{\vdots} & \boxed{\vdots} & & \\ & & & & & & & \boxed{1} & \boxed{1} & & \\ & & & & & & & \boxed{0} & \boxed{0} & & \\ & & & & & & & \boxed{\vdots} & \boxed{\vdots} & & \\ n & & & & & & & \boxed{0} & \boxed{0} & & \end{pmatrix} \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} i' \\ \\ \\ \\ \\ \\ \\ \\ \\ j' \end{array}$$

Note, that $j' := i' - l$ and $j' \geq 1$. Since, A' is symmetric we have the two possibilities a) $a'_{ij} = 0$ and $a'_{ji} = 0$ and b) $a'_{ij} = 1$ and $a'_{ji} = 1$. Hence, we have at least three possible vertical and three possible (i, j) -shifts and so $l \geq 2$. In both cases do exist $\binom{2i'-l}{i'}$ digraph realizations for S' and $\binom{2i'-l}{i'-l+1}$ digraph realizations of S . With Proposition 3 we have $\binom{2i'-l}{i'-l+1} > \binom{2i'-l}{i'}$.

Since, there exists a bijective mapping between symmetric digraph sequences and graph sequences, case 1 and case 2 lead to $N(s) > N(s')$. \square

Corollary 6. *Let $s := (a_1, \dots, a_n)$ be a decreasingly sorted integer sequence and $s' := (a'_1, \dots, a'_n)$ an integer sequence with $s \prec s'$ where $s \neq s'$. Then $N(s) > N(s')$.*

Proof. There exists at least one transfer path $s' := s^1, \dots, s^r =: s$ with $s^{i+1} \prec s^i$ and $t_{k,l}(s^i) = s^{i+1}$ by Theorem 5. We show by induction with respect to r the correctness of the claim. For $r = 2$ we apply Theorem 15 and get $N(s') < N(s)$. Let us now assume $r > 2$. We consider the transfer path s^2, \dots, s . With our induction hypotheses we can conclude $N(s^2) < N(s)$. For s' and s^2 we apply again Theorem 15. This yields $N(s') \leq N(s^2) < N(s)$. \square

Similar to the loop-digraph case a transfer path for graph sequences is strongly monotone with respect to the number of realizations. This is not always the case for digraph sequences. So we can find a similar result for a special type of transfer paths like in Corollary 1 leading to an exponential lower bound for the number of graph realizations. Lastly, we again find a result for minconvex graph sequences.

Corollary 7. *Let $s := (a_1, \dots, a_n)$ be an arbitrary graph sequence with $\sum_{i=1}^n a_i = m$, $a_1 \geq \dots \geq a_n$. Then we find for the minconvex sequence α that $N(\alpha) \geq N(s)$.*

Proof. With Theorem 11 we find that $\alpha \prec s$. Applying Corollary 6 we get $N(\alpha) \geq N(s)$. \square

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